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GENERALIZATION OF THE POISEUILLE LAW BASED ON THE CONSTITUTIVE RHEOLOGICAL RELATION FOR POLYMERIC LIQUIDS

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The method of perturbations of the small parameter determining the anisotropy of the properties of linear polymers is used to determine the velocity profile and rate for steady flow in a round tube. It is shown that for the four-parameter rheological model considered, the stress state of the Poiseuille flow along with the tangential shear stress is characterized by the first and second differences of normal stresses.

At present, polymers are finding ever increasing use in the extracting and processing industries. The methods of producing articles from polymer materials are continuously changing. Therefore, studying the technologies of polymer processing is an important practical problem. Solution of this problem requires a mathematical formulation of the laws of behavior of polymer fluids in various units of technological equipment. Polymer fluids (polymer solutions and melts) have a complex internal structure, and, depending on deformation conditions, they can show nonlinear-viscous properties, partially store energy delivered from outside, and relax stresses.

In the case of flows of solutions and melts of linear polymers, the law governing their behavior is formulated as a rheological constitutive relation describing the nonlinearly viscoelastic properties of polymer fluids. However, a consistent rheological constitutive relation describing various regimes of polymer flows has not been obtained. It is important, therefore, to verify the agreement between existing rheological models and real polymer flows by calculations of test examples of various complexity. The most simple are viscosimetric flows: pure shear and uniaxial tension. A rheological model obtained as a zero approximation of small molecular parameters of the microviscoelasticity theory of linear polymers is proposed and studied in [1–3]. The equations of this model have the form

$$\rho\Big(\frac{\partial v_i}{\partial t}+v_k\nu_{ik}\Big)=\frac{\partial\sigma_{ik}}{\partial x_k}+F_k,\qquad \frac{\partial v_k}{\partial x_k}=0,$$

$$\frac{d\xi_{ij}}{dt}-\nu_{ik}\xi_{kj}-\nu_{jk}\xi_{ki}=-\frac{1}{\tau}\Big(\xi_{ij}-\frac{1}{3}\delta_{ij}\Big)-\frac{3\beta}{\tau_0}\Big(\xi_{ik}-\frac{1}{3}\delta_{ik}\Big)\Big(\xi_{kj}-\frac{1}{3}\delta_{kj}\Big),\tag{1}$$

$$\sigma_{ij}=-p\delta_{ij}+3\frac{\eta_0}{\tau_0}(\xi_{ij}-\frac{1}{3}\delta_{ij});$$

$$\tau=\frac{\tau_0}{1+(æ-\beta)D},\qquad D=\frac{\tau_0}{\eta_0}(\sigma_{ii}+3p).\tag{2}$$

Here p is the pressure, t is the time, σ_{ij} and $\nu_{ik}=\partial v_i/\partial x_k$ are the stress and velocity-gradient tensors, ξ_{ik} is a tensorial internal thermodynamic parameter, ρ is the density of the medium, η_0 and τ_0 are the initial shear viscosity and relaxation time, æ and β are anisotropy mobility parameters that take into account [1–3] the effect of the volume and shape of macromolecular beads on the dynamics of a macromolecule, and D is a first invariant of the tensor of additional stresses.

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It has been shown [1–3] that the theoretical dependences of stationary viscosimetric functions for simple shear are in good agreement with experimental data for a number of solutions and melts of linear polymers. Therefore, in [1–3], the rheological model (1) is recommended as a basis for engineering calculations. This, however, requires consideration of more complex flows than the ones realized in viscosimeters, for example, stationary flow in a smooth round tube under the action of a constant drop in pressure. In this case, system (1) is conveniently written in cylindrical coordinates. Using the formulas for covariant differentiation of tensor components [4] in the stationary case, we have

$$\frac{\partial p}{\partial r}=\frac{3\eta_0}{\tau_0}\Big(\frac{\partial\xi_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\xi_{r\varphi}}{\partial\varphi}+\frac{\xi_{rr}-\xi_{\varphi\varphi}}{r}+\frac{\partial\xi_{rz}}{\partial z}\Big),\quad \frac{1}{r}\frac{\partial p}{\partial\varphi}=\frac{3\eta_0}{\tau_0}\Big(\frac{\partial\xi_{r\varphi}}{\partial r}+\frac{2}{r}\xi_{r\varphi}+\frac{1}{r}\frac{\partial\xi_{\varphi\varphi}}{\partial\varphi}+\frac{\partial\xi_{\varphi z}}{\partial z}\Big),$$

$$\frac{\partial p}{\partial z}=\frac{3\eta_0}{\tau_0}\Big(\frac{\partial\xi_{rz}}{\partial r}+\frac{\xi_{rz}}{r}+\frac{1}{r}\frac{\partial\xi_{\varphi z}}{\partial\varphi}+\frac{\partial\xi_{rz}}{\partial z}\Big),\qquad \frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi}+\frac{\partial v_z}{\partial z}=0,$$

$$-2\Big(\frac{\partial v_r}{\partial r}\xi_{rr}+\frac{1}{r}\frac{\partial v_r}{\partial\varphi}\xi_{r\varphi}+\frac{\partial v_r}{\partial z}\xi_{rz}\Big)=-\frac{1}{\tau}\Big(\xi_{rr}-\frac{1}{3}\Big)-\frac{3\beta}{\tau_0}\Big[\Big(\xi_{rr}-\frac{1}{3}\Big)^2+\xi_{r\varphi}^2+\xi_{rz}^2\Big],$$

$$\Big(\frac{v_\varphi}{r}-\frac{\partial v_\varphi}{\partial r}\Big)\xi_{rr}+\frac{\partial v_z}{\partial z}\xi_{r\varphi}-\frac{\partial v_\varphi}{\partial z}\xi_{rz}-\frac{1}{3}\frac{\partial v_r}{\partial\varphi}\xi_{\varphi\varphi}-\frac{\partial v_r}{\partial z}\xi_{\varphi z}$$

$$=-\frac{1}{\tau}\xi_{r\varphi}-\frac{3\beta}{\tau_0}\Big[\Big(\xi_{rr}-\frac{1}{3}\Big)\xi_{r\varphi}+\xi_{r\varphi}\Big(\xi_{\varphi\varphi}-\frac{1}{3}\Big)+\xi_{rz}\xi_{\varphi z}\Big],$$

$$-\Big[\Big(\frac{\partial v_r}{\partial r}+\frac{\partial v_z}{\partial z}\Big)\xi_{rz}+\frac{\partial v_z}{\partial r}\xi_{rr}+\frac{1}{r}\frac{\partial v_z}{\partial\varphi}\xi_{r\varphi}+\frac{1}{r}\frac{\partial v_r}{\partial\varphi}\xi_{\varphi z}+\frac{\partial v_r}{\partial z}\xi_{zz}\Big] \tag{3}$$

$$=-\frac{1}{\tau}\xi_{rz}-\frac{3\beta}{\tau_0}\Big[\Big(\xi_{rr}-\frac{1}{3}\Big)\xi_{rz}+\xi_{r\varphi}\xi_{\varphi z}+\xi_{rz}\Big(\xi_{zz}-\frac{1}{3}\Big)\Big],$$

$$2\Big[\Big(\frac{v_\varphi}{r}-\frac{\partial v_\varphi}{\partial r}\Big)\xi_{r\varphi}-\frac{1}{r}\Big(\frac{\partial v_\varphi}{\partial\varphi}+v_r\Big)\xi_{\varphi\varphi}-\frac{\partial v_\varphi}{\partial z}\xi_{z\varphi}\Big]=-\frac{1}{\tau}\Big(\xi_{\varphi\varphi}-\frac{1}{3}\Big)-\frac{3\beta}{\tau_0}\Big[\xi_{r\varphi}^2+\Big(\xi_{\varphi\varphi}-\frac{1}{3}\Big)^2+\xi_{\varphi z}^2\Big],$$

$$-2\Big(\frac{\partial v_z}{\partial r}\xi_{rz}+\frac{1}{r}\frac{\partial v_z}{\partial\varphi}\xi_{\varphi z}+\frac{\partial v_z}{\partial z}\xi_{zz}\Big)=-\frac{1}{\tau}\Big(\xi_{zz}-\frac{1}{3}\Big)-\frac{3\beta}{\tau_0}\Big[\xi_{rz}^2+\xi_{\varphi z}^2+\Big(\xi_{zz}-\frac{1}{3}\Big)^2\Big],$$

$$\Big(\frac{v_\varphi}{r}-\frac{\partial v_\varphi}{\partial r}\Big)\xi_{rz}+\frac{\partial v_r}{\partial r}\xi_{\varphi z}-\frac{\partial v_z}{\partial z}\xi_{zz}-\frac{\partial v_z}{\partial r}\xi_{r\varphi}-\frac{1}{r}\frac{\partial v_z}{\partial\varphi}\xi_{\varphi\varphi}$$

$$=-\frac{1}{\tau}\xi_{\varphi z}-\frac{3\beta}{\tau_0}\Big[\xi_{r\varphi}\xi_{rz}+\Big(\xi_{\varphi\varphi}-\frac{1}{3}\Big)\xi_{\varphi z}+\xi_{\varphi z}\Big(\xi_{zz}-\frac{1}{3}\Big)\Big].$$

Here the first invariant of the tensor of additional stresses is given by the formula

$$D=3(\xi_{rr}+\xi_{\varphi\varphi}+\xi_{zz}-1). \tag{4}$$

We note that nonstationary equations, which are also of practical interest, can be obtained from (3) by supplementing appropriate terms from [5]. Further we assume that the mass-force vector $\boldsymbol{F}$ is equal to zero.

In the case $\beta=0$, solution (3), (4) reduces to the results obtained in [5] using the structural-phenomenological model of Pokrovskii. We shall seek an axisymmetric solution of system (2)–(4) that does not depend on the z coordinate. Since, according to the estimates in [2, 3], it is assumed that $\beta\ll 1$, we seek this solution with first-order accuracy with respect to β:

$$v_r=v_r^0+\beta v_r',\qquad v_\varphi=v_\varphi^0+\beta v_\varphi',\qquad v_z=v_z^0+\beta v_z',\qquad \xi_{rr}=\xi_{rr}^0+\beta\xi_{rr}',\qquad \xi_{r\varphi}=\xi_{r\varphi}^0+\beta\xi_{r\varphi}',$$

$$\xi_{rz}=\xi_{rz}^0+\beta\xi_{rz}',\qquad \xi_{\varphi\varphi}=\xi_{\varphi\varphi}^0+\beta\xi_{\varphi\varphi}',\qquad \xi_{\varphi z}=\xi_{\varphi z}^0+\beta\xi_{\varphi z}',\qquad \xi_{zz}=\xi_{zz}^0+\beta\xi_{zz}'.$$

Here the superscript 0 corresponds to the zero approximation for β, and the prime corresponds to the first approximation. We note that a zero-approximation solution is obtained in [5] and it has the form

$$v_z=u(r)=\frac{A}{4\eta_0}(R^2-r^2)\Big[1+\frac{æA^2\tau_0^2}{4\eta_0^2}(R^2+r^2)\Big],\qquad \xi_{rz}^0=\xi_1(r)=-\frac{\tau_0}{3\eta_0}\frac{A}{2}r,$$

$$v_r^0 = v_\varphi^0 = \xi_{r\varphi}^0 = \xi_{z\varphi}^0 = 0, \qquad \frac{\partial p}{\partial z} = -A = \text{const}, \qquad \xi_{rr}^0 = \xi_{\varphi\varphi}^0 = \frac{1}{3}, \qquad \xi_{zz}^0 = \frac{1}{3}(1 + D),$$

Consequently,

$$v_r = \beta v_r', \quad v_\varphi = \beta v_\varphi', \quad v_z = u(r) + \beta v_z', \quad D = D(r), \quad \xi_{rr} = \frac{1}{3} + \beta\xi_{rr}', \quad \xi_{r\varphi} = \beta\xi_{r\varphi}',$$
$$\xi_{rz} = \xi_1(r) + \beta\xi_{rz}', \qquad \xi_{\varphi\varphi} = \frac{1}{3} + \beta\xi_{\varphi\varphi}', \qquad \xi_{\varphi z} = \beta\xi_{\varphi z}', \qquad \xi_{zz} = \frac{1}{3}(1 + D) - \beta(\xi_{rr}' + \xi_{\varphi\varphi}'), \tag{5}$$
$$\frac{\partial p}{\partial z} = -A = \text{const}.$$

With allowance for (5), we write system (3), (4) with first-order accuracy with respect to the anisotropy coefficient β:

$$\frac{3\eta_0}{\tau_0}\Big(\frac{d\xi_{rr}'}{dr} + \frac{\xi_{rr}' - \xi_{\varphi\varphi}'}{r}\Big) = \frac{\partial p}{\partial r}; \tag{6}$$

$$\frac{d\xi_{r\varphi}'}{dr} + \frac{2}{r}\,\xi_{r\varphi}' = 0; \tag{7}$$

$$\frac{3\eta_0}{\tau_0}\Big(\frac{d\xi_{rz}}{dr} + \frac{\xi_{rz}}{r}\Big) = -A; \tag{8}$$

$$\frac{dv_r'}{dr} + \frac{v_r'}{r} = 0; \tag{9}$$

$$\frac{1}{\tau}\,\xi_{rr}' + \frac{3}{\tau_0}\,\xi_1^2(r) = 0; \tag{10}$$

$$\frac{1}{3}\Big(\frac{v_\varphi'}{r} - \frac{dv_\varphi'}{dr}\Big) = -\frac{1}{\tau}\,\xi_{r\varphi}'; \tag{11}$$

$$-\beta\,\frac{dv_r'}{dr}\,\xi_1(r) - \frac{dv_z}{dr}\,\xi_{rr} = -\frac{1}{\tau}\,\xi_{rz} - \frac{3\beta}{\tau_0}\,\xi_1(r)\Big(\xi_{zz} - \frac{1}{3}\Big); \tag{12}$$

$$\frac{2}{3r}\,v_r' = \frac{1}{\tau}\,\xi_{\varphi\varphi}'; \tag{13}$$

$$-2\frac{dv_z}{dr}\xi_{rz} = -\frac{1}{\tau}\Big(\xi_{zz} - \frac{1}{3}\Big) - \frac{3\beta}{\tau_0}\Big[\xi_1^2(r) + \Big(\xi_{zz} - \frac{1}{3}\Big)^2\Big], \quad \Big(\frac{v_\varphi'}{r} - \frac{dv_\varphi'}{dr}\Big)\xi_1(r) - \frac{dv_z}{dr}\xi_{r\varphi}' = -\frac{1}{\tau}\xi_{\varphi z}'. \tag{14}$$

Since from Eq. (7) it follows that $\xi_{r\varphi}' = C/r^2$, we obtain v_φ' from (11): $v_\varphi' = Kr - 3C/(2\tau r)$.

The continuity equation (9) gives $v_r' = M/r$. Then, formula (13) is written as $\xi_{\varphi\varphi}' = 2\tau M/(3r^2)$.

The constants C, K, and M are determined from the boundary conditions. We obtain $\xi_{r\varphi}' = \xi_{\varphi\varphi}' = \xi_{\varphi z}' = v_r' = v_\varphi' = 0$.

From the equation of motion (8) we find the following expression for ξ_{rz}:

$$\xi_{rz} = \frac{\tau_0}{3\eta_0}\Big(-\frac{A}{2}\,r + \frac{B}{r}\Big),$$

i.e. $\xi_{rz} = \xi_1(r)$ and $\xi_{rz}' = 0$. From the condition of limited shear strain on the symmetry axis of the tube, we have $B = 0$.

Thus,

$$\xi_1(r) = -\frac{A\tau_0}{6\eta_0}\,r. \tag{15}$$

From (10) we obtain the formula

$$\xi_{rr}' = -\frac{3\tau}{\tau_0}\,\xi_1^2, \qquad \xi_{rr} = \frac{1}{3} - \beta\,\frac{3\tau}{\tau_0}\,\xi_1^2. \tag{16}$$

The relaxations equations (12) and (14) lead to the relations

$$-\frac{dv_z}{dr}\xi_{rr} = -\frac{1}{\tau}\xi_1(r) - \frac{3\beta}{\tau_0}\xi_1(r)\Big(\xi_{zz} - \frac{1}{3}\Big), \quad -2\frac{dv_z}{dr}\xi_1(r) = -\frac{1}{\tau}\Big(\xi_{zz} - \frac{1}{3}\Big) - \frac{3\beta}{\tau_0}\Big[\xi_1^2(r) + \Big(\xi_{zz} - \frac{1}{3}\Big)^2\Big],$$

which, with allowance for $\tau = \tau_0$, are written with zero-order accuracy with respect to æ and β in the form

$$\frac{dv_z}{dr} = \frac{1}{\tau_0}\frac{\xi_1(r)}{\xi_{rr}}(1+\beta D), \quad \frac{dv_z}{dr} = \frac{1}{2\tau_0}\frac{1}{\xi_1(r)}\Big(\frac{D}{3} - \beta\xi'_{rr} + 3\beta\xi_1^2(r) + \frac{\beta D^2}{3}\Big).$$

Thus, we obtain the following formula for v_z

$$-v_z = \frac{1}{\tau_0}\int_r^R \frac{\xi_1(r)}{\xi_{rr}}\Big[1 + \beta D(r)\Big]dr \qquad (17)$$

and the quadratic equation for D

$$\beta D^2 + (1 + 3\beta\xi'_{rr} - 18\xi_1^2)D + 9\beta\xi_1^2 - 3\beta\xi'_{rr} - 18\xi_1^2 = 0,$$

whose the solution for small β has the form

$$D = 18\,\xi_1^2. \qquad (18)$$

Substituting (15), (16), and (18) into formula (17), we obtain

$$v_z = \frac{A}{4\eta_0}(R^2 - r^2)\Big[1 + \frac{æA^2\tau_0^2}{4\eta_0^2}(R^2 + r^2) + \beta\,\frac{A^2\tau_0^2}{24\eta_0^2}(R^2 + r^2)\Big]. \qquad (19)$$

Hence it follows that the experimentally observed [6] deflection of the velocity profile from a parabola is related to the rheological parameters æ and β. Calculations using formula (19) show that as β and æ are varied from 0 to 1, the deflection of the velocity profile from a parabolic one reaches 5–10%. Thus, fairly accurate measurements of the velocity profile can be the basis for determining the parameters of the model æ and β.

From Eq. (6) we establish the presence of nonzero drop in pressure in the radial direction:

$$\frac{\partial p}{\partial r} = -\frac{3A^2\tau_0\beta}{4\eta_0}r. \qquad (20)$$

However, the drop in pressure doe not give rise to flow in this direction. Probably, with increase in the pressure gradient in the axial direction, the stationary solution of the equations becomes unstable, and this gives rise to secondary flows. Such calculation involves considerable difficulties even for media described by the Navier–Stokes equations.

The pressure $p(r,z) = p(z) + \beta p(r)$ is obtained by integration of (6) and (20):

$$p = -Az - \beta\,\frac{3A^2\tau_0}{8\eta_0}r^2 + p_0. \qquad (21)$$

The expression for the volumetric flow rate $Q = 2\pi\rho\int_0^R rv_z\,dr$ takes the form

$$Q = \frac{\pi\rho A}{8\eta_0}R^4\Big(1 + \frac{A^2\tau_0^2æ}{3\eta_0^2}R^2 + \beta\,\frac{A^2\tau_0^2}{18\eta_0^2}R^2\Big). \qquad (22)$$

For the class of flows considered, the first and second differences of normal stresses are equal to

$$\sigma_1 \equiv \sigma_{zz} - \sigma_{rr} = \frac{A^2\tau_0}{2\eta_0}r^2 + \beta\,\frac{A^2\tau_0}{2\eta_0}r^2; \qquad (23)$$

$$\sigma_2 \equiv \sigma_{rr} - \sigma_{\varphi\varphi} = -\beta\,\frac{A^2\tau_0}{4\eta_0}r^2. \qquad (24)$$

From (23) and (24) it follows that $\sigma_2/\sigma_1 = -\beta/2$ with first-order accuracy with respect to β, i.e., it is the same as in the case of stationary Couette flow [1–3].

The terms without β and æ in (19) and (22) are known for a viscous liquid, and the terms with β and æ introduce a correction for non-Newtonian viscoelastic behavior. In this case, from (19) and (22) we obtain formulas for $\beta = 0$ that were reported previously in [5]. The relations obtained can be used to determine the material constants η_0, τ_0, β, and æ from the experimental dependence of the flow rate on the applied pressure gradient.

In addition Eqs. (5), (15), (16), (18)–(21), (23), (24) can be used in calculations of more complex flows than Poiseuille flow, for example, for forced flow between two coaxial cylinders one of which (for example, the outer one) is immovable, and the inner cylinder moves along the axis at a constant velocity, or for flows that arise in a ring channel formed by fixed concentric cylinders under the action of specified pressure gradient. In this case, all constants can be found from the boundary conditions if it is assumed, as before, that the motion os steady and rectilinear.

REFERENCES

1. G. V. Pyshnograi, V. N. Pokrovskii, Yu. G. Yanovskii, et al., "Constitutive equation of nonlinear viscoelastic polymer media in a zero approximation of the parameters of molecular theory and an implication for shear and tension," *Dokl. Ross. Akad. Nauk*, **339**, No. 5, 612–615 (1994).
2. G. V. Pyshnograio, "An initial approximation in the theory of microviscoelasticity of linear polymers and associated nonlinear effects," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 1, 145–151 (1996).
3. Yu. A. Altukhov and G. V. Pyshnograi, "Microstructural approach in the theory of flow of linear polymers and associated nonlinear effects," *Vysokomol. Soed., Ser. A.*, **38**, No. 7, 1185–1193 (1996).
4. L. I. Sedov, *Mechanics of Continuous Media* [in Russian], Vol. 1, Nauka, Moscow (1983).
5. V. B. Érenburg and V. N. Pokrovskii, "Nonuniform shear flows of linear polymers," *Inzh. Fiz. Zh.*, **41**, No. 3, 449–456 (1981).
6. Chang Dei Han, *Rheology in Polymer Processing* [Russian translation], Khimiya, Moscow (1979).